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On the square integrability of the q-Hermite functions

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Abstract

Overlap integrals over the full real line $-\infty<x<\infty$ for a family of the q-Hermite functions $H_n(\sin\kappa x|q)\mathrm{e}^{-x^2/2}$, $0<q=\mathrm{e}^{-2\kappa^2}<1$ are evaluated. In particular, an explicit form of the squared norms for these q-extensions of the Hermite functions (or the wave functions of the linear harmonic oscillator in quantum mechanics) is obtained. The classical Fourier–Gauss transform connects the q-Hermite functions with different values $0<q<1$ and $q>1$ of the parameter q. An explicit expansion of the q-Hermite polynomials $H_n(\sin\kappa x|q)$ in terms of the Hermite polynomials $H_n(x)$ emerges as a by-product.

1. Introduction

The Hermite functions

$$\psi_n(\xi):=[\sqrt{\pi}2^n n!]^{-1/2}H_n(\xi)\mathrm{e}^{-\xi^2/2}, \tag{1.1}$$

where $H_n(\xi)$ are the classical Hermite polynomials, are of great mathematical interest as an explicit example of an orthonormal and complete system in the Hilbert space $L^2(\mathbb{R})$ of square-integrable functions with respect to the full real line $-\infty<\xi<\infty$ [1]. In mathematical physics they are known to represent solutions of the linear harmonic oscillator problem, which plays a very important role in quantum mechanics. In what follows, we attempt to study in detail some particular q-generalization of the Hermite functions (1.1).

2. Overlap integrals and squared norms

Let us consider a family of q-Hermite functions

$$\psi_n(\xi|q):=c_n(q)H_n(\sin\kappa\xi|q)\mathrm{e}^{-\xi^2/2},\quad 0<q=\mathrm{e}^{-2\kappa^2}<1, \tag{2.1}$$

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where the normalization constant $c_n(q)=[\sqrt{\pi}(q;q)_n]^{-1/2}$ and the q-shifted factorial $(q;q)_n$ is defined as $(z;q)_0=1$ and $(z;q)_n=\prod_{k=0}^{n-1}(1-zq^k)$, $n=1,2,3\ldots$ (throughout this paper, we will employ the standard notations of q-special functions, see [2] or [3]). The continuous q-Hermite polynomials $H_n(x|q)$ in (2.1) are those q-extensions of the ordinary Hermite polynomials $H_n(x)$, which satisfy the three-term recurrence relation

$$H_{n+1}(x|q)=2xH_n(x|q)-(1-q^n)H_{n-1}(x|q), \quad n=0,1,2,\ldots, \tag{2.2}$$

with the initial condition $H_0(x|q)=1$ [4]. Their explicit form is given by the Fourier expansion

$$H_m(\sin\kappa\xi|q)=\imath^m\sum_{n=0}^{m}(-1)^n\begin{bmatrix} m \\ n \end{bmatrix}_q \mathrm{e}^{\imath(2n-m)\kappa\xi}, \tag{2.3}$$

where $\begin{bmatrix} m \\ n \end{bmatrix}_q$ is the q-binomial coefficient,

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q;q)_m}{(q;q)_n(q;q)_{m-n}} = \begin{bmatrix} m \\ m-n \end{bmatrix}_q. \tag{2.4}$$

The q-Hermite polynomials (2.3) are solutions of the difference equation

$$\left[\mathrm{e}^{\mathrm{i}\kappa s}\exp\left(-\mathrm{i}\kappa\frac{\mathrm{d}}{\mathrm{d}s}\right)+\mathrm{e}^{-\mathrm{i}\kappa s}\exp\left(\mathrm{i}\kappa\frac{\mathrm{d}}{\mathrm{d}s}\right)\right]H_n(\sin\kappa s|q)=2q^{-n/2}\cos\kappa s\,H_n(\sin\kappa s|q). \tag{2.5}$$

It is easy to verify that $\lim_{q\to1}\kappa^{-2n}(q;q)_n=2^n n!$. Therefore it follows from the recurrence relation (2.2), that

$$\lim_{q\to1}\kappa^{-n}H_n(\sin\kappa\xi|q)=H_n(\xi). \tag{2.6}$$

Thus, the normalization in (2.1) is chosen so that when the limit $q\to1$ is taken they coincide with the wave functions $\psi_n(\xi)$ of the linear harmonic oscillator in quantum mechanics, i.e.

$$\psi_n(\xi|1):=\lim_{q\to1}\psi_n(\xi|q)=\psi_n(\xi). \tag{2.7}$$

As it is well known, the wave functions $\psi_n(\xi)$ satisfy the orthogonality relation

$$\int_{-\infty}^{\infty}\psi_m(\xi)\psi_n(\xi)\,\mathrm{d}\xi=\delta_{mn} \tag{2.8}$$

and may serve as a basis in the Hilbert space $L_2(\mathbb{R})$ of square-integrable functions with respect to $\mathrm{d}\xi$.

We evaluate first the corresponding integral

$$I_{m,n}(q):=\int_{-\infty}^{\infty}\psi_m(\xi|q)\psi_n(\xi|q)\,\mathrm{d}\xi=I_{n,m}(q) \tag{2.9}$$

for the q-Hermite functions (2.1). Since $H_n(-x|q)=(-1)^n H_n(x|q)$ by definition, the functions $\psi_m(\xi|q)$ and $\psi_n(\xi|q)$ of the opposite parities $(m-n=2k+1,\ k=0,1,2,\ldots)$ are orthogonal and nontrivial integrals in (2.9) are

$$I_{n,n+2k}(q)=[\pi(q;q)_n(q;q)_{n+2k}]^{-1/2}\int_{-\infty}^{\infty} H_n(\sin\kappa\xi|q)H_{n+2k}(\sin\kappa\xi|q)\mathrm{e}^{-\xi^2}\,\mathrm{d}\xi. \tag{2.10}$$

Using the Rogers linearization formula [5]

$$H_m(x|q)H_n(x|q)=\sum_{k=0}^{m}\frac{(q;q)_n}{(q;q)_{n-m+k}}\begin{bmatrix} m \\ k \end{bmatrix}_q H_{n-m+2k}(x|q),\quad m\leqslant n, \tag{2.11}$$

for the q-Hermite polynomials (2.3), one can represent (2.10) as

$$I_{n,n+2k}(q)=\frac{(q^{n+1};q)_{2k}^{1/2}}{\sqrt{\pi}}\sum_{l=0}^{n}(q;q)_{2k+l}^{-1}\begin{bmatrix} n \\ l \end{bmatrix}_q\int_{-\infty}^{\infty}H_{2k+2l}(\sin\kappa\xi|q)\mathrm{e}^{-\xi^2}\,\mathrm{d}\xi. \tag{2.12}$$

It remains only to substitute the explicit form of the q-Hermite polynomials (2.3) into (2.12) and to evaluate the integral with respect to the variable ξ by the aid of the well-known integral transform

$$\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{2\imath xy-x^2}=\sqrt{\pi}\mathrm{e}^{-y^2}. \tag{2.13}$$

The result is

$$\begin{aligned} I_{n,n+2k}(q)=(q^{n+1};q)_{2k}^{1/2}\sum_{l=0}^{n}(-1)^{k+l}\begin{bmatrix} n \\ l \end{bmatrix}_q\frac{q^{(k+l)^2/2}}{(q;q)_{2k+l}} \\ \times\sum_{j=0}^{2(k+l)}(-1)^j\begin{bmatrix} 2(k+l) \\ j \end{bmatrix}_q q^{j^2/2-j(k+l)}. \end{aligned} \tag{2.14}$$

The sum over j in (2.14) gives the factor $(q^{1/2-k-l};q)_{2(k+l)}$ because of the Gauss identity

$$(z;q)_n=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2}(-z)^k. \tag{2.15}$$

In view of the formulas (see, for example, [2] or [3])

$$(z;q)_{n+k}=(z;q)_n(zq^n;q)_k, \tag{2.16}$$
$$(zq^{-n};q)_n=q^{-n(n+1)/2}(-z)^n(q/z;q)_n, \tag{2.17}$$

this factor is equal to

$$(q^{1/2-k-l};q)_{2(k+l)}=(-1)^{k+l}q^{-(k+l)^2/2}(q^{1/2};q)_{k+l}^2. \tag{2.18}$$

Now substituting (2.18) into (2.14), yields

$$I_{n,n+2k}(q)=(q^{n+1};q)_{2k}^{1/2}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\frac{(q^{1/2};q)_{k+l}^2}{(q;q)_{2k+l}}. \tag{2.19}$$

Since $(q;q)_{2k+l}=(q;q)_{2k}(q^{2k+1};q)_l$ by (2.16) and

$$(q;q)_{n-m}=(-1)^m q^{m(m-1)/2-mn}\frac{(q;q)_n}{(q^{-n};q)_m}, \tag{2.20}$$

one can express (2.19) through the basic hypergeometric series ${}_3\phi_1$:

$$I_{n,n+2k}(q)=\frac{(q^{1/2};q)_k^2}{(q;q)_{2k}}(q^{n+1};q)_{2k}^{1/2}\,{}_3\phi_1(q^{-n},q^{k+1/2},q^{k+1/2};q^{2k+1};q,q^n). \tag{2.21}$$

A particular case of (2.21) with $k=0$,

$$I_{n,n}(q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{1/2};q)_k^2}{(q;q)_k}={}_3\phi_1(q^{-n},q^{1/2},q^{1/2};q;q,q^n), \tag{2.22}$$

represents the squared norm of the q-Hermite function $\psi_n(\xi|q)$. As is evident from (2.22), $I_{n,n}(q)$ is finite and positive for all the values of $q\in(0,1)$.

3. Orthogonalization

It is clear that the q-Hermite functions (2.1) are linearly independent, for they are expressed through the q-Hermite polynomials of different order (multiplied by the common exponential factor $e^{-\xi^2/2}$). Therefore, once the overlap integrals (2.19) for them are known, the system $\{\psi_n(\xi|q)\}$ can be orthogonalized by the formation of suitable linear combinations. Since the subsequences $\{\psi_{2k}(\xi|q)\}$ and $\{\psi_{2k+1}(\xi|q)\}$, $k=0,1,2,\ldots$, are mutually orthogonal by definition, one needs to form such combinations for the even and odd functions separately. In other words, if we define (see [6, p. 154])

$$\tilde{\psi}_{2k}(\xi|q)=\begin{vmatrix} I_{0,0}(q) & I_{0,2}(q) & \cdots & I_{0,2k}(q) \\ I_{2,0}(q) & I_{2,2}(q) & \cdots & I_{2,2k}(q) \\ \vdots & \vdots & \vdots & \vdots \\ I_{2k-2,0}(q) & I_{2k-2,2}(q) & \cdots & I_{2k-2,2k}(q) \\ \psi_0(\xi|q) & \psi_2(\xi|q) & \cdots & \psi_{2k}(\xi|q) \end{vmatrix}$$

$$=e^{-\xi^2/2}\begin{vmatrix} I_{0,0}(q) & I_{0,2}(q) & \cdots & I_{0,2k}(q) \\ I_{2,0}(q) & I_{2,2}(q) & \cdots & I_{2,2k}(q) \\ \vdots & \vdots & \vdots & \vdots \\ I_{2k-2,0}(q) & I_{2k-2,2}(q) & \cdots & I_{2k-2,2k}(q) \\ c_0(q)H_0(\sin\kappa\xi|q) & c_2(q)H_2(\sin\kappa\xi|q) & \cdots & c_{2k}(q)H_{2k}(\sin\kappa\xi|q) \end{vmatrix}, \tag{3.1}$$

then $\{\tilde{\psi}_{2k}(\xi|q)\}$ is an orthogonal system, for (3.1) is orthogonal to $\psi_0(\xi|q),\psi_2(\xi|q),\ldots,\psi_{2k-2}(\xi|q)$ and hence to $\tilde{\psi}_{2n}(\xi|q)$ for all $n<k$.

Similarly, for the odd q-Hermite functions the appropriate linear combinations are

$$\tilde{\psi}_{2k+1}(\xi|q) = \begin{vmatrix} I_{1,1}(q) & I_{1,3}(q) & \cdots & I_{1,2k+1}(q) \\ I_{3,1}(q) & I_{3,3}(q) & \cdots & I_{3,2k+1}(q) \\ \vdots & \vdots & \vdots & \vdots \\ I_{2k-1,1}(q) & I_{2k-1,3}(q) & \cdots & I_{2k-1,2k+1}(q) \\ \psi_1(\xi|q) & \psi_3(\xi|q) & \cdots & \psi_{2k+1}(\xi|q) \end{vmatrix}$$

$$= \mathrm{e}^{-\xi^2/2} \begin{vmatrix} I_{1,1}(q) & I_{1,3}(q) & \cdots & I_{1,2k+1}(q) \\ I_{3,1}(q) & I_{3,3}(q) & \cdots & I_{3,2k+1}(q) \\ \vdots & \vdots & \vdots & \vdots \\ I_{2k-1,1}(q) & I_{2k-1,3}(q) & \cdots & I_{2k-1,2k+1}(q) \\ c_1(q)H_1(\sin\kappa\xi|q) & c_3(q)H_3(\sin\kappa\xi|q) & \cdots & c_{2k+1}(q)H_{2k+1}(\sin\kappa\xi|q) \end{vmatrix}. \tag{3.2}$$

A system of the functions

$$\tilde{\psi}_n(\xi|q) = c_n(q)\tilde{H}_n(\sin\kappa\xi|q)\mathrm{e}^{-\xi^2/2}, \quad n = 0, 1, 2, \ldots, \tag{3.3}$$

is thus orthogonal over the full real line $-\infty < \xi < \infty$ with respect to $\mathrm{d}\xi$. The polynomials $\tilde{H}_n(x|q)$ in (3.3) are linear combinations of the q-Hermite polynomials (2.3) of the form

$$\tilde{H}_n(x|q) = \sum_{k=0}^{n} \alpha_{n,k}(q)H_k(x|q). \tag{3.4}$$

From the second determinants in (3.1) and (3.2) it follows that the connection coefficients $\alpha_{n,k}(q)$ in (3.4) are equal to

$$\alpha_{2k,2j}(q) = (-1)^{k+j+1} \left[\frac{(q;q)_{2k}}{(q;q)_{2j}}\right]^{1/2}$$

$$\times \begin{vmatrix} I_{0,0}(q) & I_{0,2}(q) & \cdots & I_{0,2j-2}(q) & I_{0,2j+2}(q) & \cdots & I_{0,2k}(q) \\ I_{2,0}(q) & I_{2,2}(q) & \cdots & I_{2,2j-2}(q) & I_{2,2j+2}(q) & \cdots & I_{2,2k}(q) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I_{2k-2,0}(q) & I_{2k-2,2}(q) & \cdots & I_{2k-2,2j-2}(q) & I_{2k-2,2j+2}(q) & \cdots & I_{2k-2,2k}(q) \end{vmatrix}, \tag{3.5}$$

$$\alpha_{2k+1,2j+1}(q) = (-1)^{k+j+1} \left[\frac{(q;q)_{2k+1}}{(q;q)_{2j+1}}\right]^{1/2}$$

$$\times \begin{vmatrix} I_{1,1}(q) & I_{1,3}(q) & \cdots & I_{1,2j-1}(q) & I_{1,2j+3}(q) & \cdots & I_{1,2k+1}(q) \\ I_{3,1}(q) & I_{3,3}(q) & \cdots & I_{3,2j-1}(q) & I_{3,2j+3}(q) & \cdots & I_{3,2k+1}(q) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I_{2k-1,1}(q) & I_{2k-1,3}(q) & \cdots & I_{2k-1,2j-1}(q) & I_{2k-1,2j+3}(q) & \cdots & I_{2k-1,2k+1}(q) \end{vmatrix}, \tag{3.6}$$

for $n = 2k$ and $n = 2k + 1$, $k = 0, 1, 2, \ldots$, respectively.

4. Fourier expansion

Having established that the q-Hermite functions $\psi_n(\xi|q)$ are square integrable, it is natural to look for their expansion in terms of the Hermite functions $\psi_n(\xi)$ (or, in other words, the linear harmonic oscillator wave functions in quantum mechanics):

$$\psi_n(\xi|q) = \sum_{k=0}^{\infty} C_{n,k}(q)\psi_k(\xi). \tag{4.1}$$

To find Fourier coefficients $C_{n,k}(q)$ of $\psi_n(\xi|q)$ with respect to the system $\{\psi_k(\xi)\}$, multiply both sides of (4.1) by $\psi_m(\xi)$ and integrate them over the variable ξ within infinite limits with the help of the orthogonality (2.8). This yields

$$C_{n,m}(q) = \int_{-\infty}^{\infty} \psi_n(\xi|q)\psi_m(\xi)\,\mathrm{d}\xi = [\pi 2^m m!(q;q)_n]^{-1/2} \int_{-\infty}^{\infty} H_n(\sin\kappa\xi|q)H_m(\xi)\mathrm{e}^{-\xi^2}\,\mathrm{d}\xi. \tag{4.2}$$

To evaluate the last integral in (4.2), substitute in the Fourier expansion (2.3) for $H_n(\sin\kappa\xi|q)$ and integrate it term by term by using the integral transform (see [6, p. 124, Eq. (23)])

$$\int_{-\infty}^{\infty} H_n(x)\mathrm{e}^{2\imath xy - x^2}\,\mathrm{d}x = \sqrt{\pi}(2\imath y)^n \mathrm{e}^{-y^2} \tag{4.3}$$

for the Hermite polynomials $H_m(\xi)$. This results in

$$C_{n,m}(q) = \frac{\imath^{n+m}\kappa^m q^{n^2/8}}{\sqrt{2^m m!(q;q)_n}} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q (2k-n)^m q^{k(k-n)/2}. \tag{4.4}$$

Reversing the order of summation in (4.4) with respect to the index k makes it evident that the Fourier coefficients $C_{n,m}(q)$ are real for $0<q<1$, namely

$$C_{n,m}(q) = \frac{\cos(n+m)\pi/2}{\sqrt{2^m m!(q;q)_n}} \kappa^m q^{n^2/8} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q (2k-n)^m q^{k(k-n)/2}. \tag{4.5}$$

Note that since the both functions $\psi_n(\xi|q)$ and $\psi_n(\xi)$ (see (1.1) and (2.1), respectively) contain the same exponential factor $\mathrm{e}^{-\xi^2/2}$, the relations (4.1) and (4.5) are equivalent to an explicit expansion

$$H_n(\sin\kappa\xi|q) = \sum_{k=0}^{\infty} a_{nk}(q)H_k(\xi) \tag{4.6}$$

of the q-Hermite polynomials in terms of ordinary Hermite polynomials. The coefficients of this expansion $a_{nk}(q)$ are real and equal to

$$a_{nk}(q) = \frac{\kappa^k q^{n^2/8}}{k!} \cos(n+k)\pi/2 \sum_{l=0}^{n} (-1)^l \begin{bmatrix} n \\ l \end{bmatrix}_q (l-n/2)^k q^{l(l-n)/2}. \tag{4.7}$$

As a consistency check, one may evaluate the sum over k in the right-hand side of (4.6) by substituting in it the coefficients $a_{nk}(q)$ from (4.7) and using the generating function

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(s) = \mathrm{e}^{2st-t^2} \tag{4.8}$$

for the ordinary Hermite polynomials $H_k(s)$. This gives indeed the explicit form (2.3) of the q-Hermite polynomials $H_n(\sin\kappa\xi|q)$ in the left-hand side of (4.6).

5. Fourier integral transform

Since the q-Hermite functions (2.1) belong to $L_2(\mathbb{R})$, one may define their Fourier transforms with the same property of the square integrability. A remarkable fact is that the classical Fourier integral transform relates the q-Hermite functions with different values $0<q<1$ and $q>1$ of the parameter q.

We remind the reader that to consider the values $1<q<\infty$ of the parameter q it is convenient to introduce [7] the q^{-1}-Hermite polynomials

$$h_n(x|q) := \imath^{-n} H_n(\imath x|q^{-1}). \tag{5.1}$$

They satisfy the three-term recurrence relation

$$h_{n+1}(x|q) = 2x h_n(x|q) + (1-q^{-n}) h_{n-1}(x|q), \quad n=0,1,2,\ldots, \tag{5.2}$$

with the initial condition $h_0(x|q)=1$. As follows from the Fourier expansion (2.3) and the inversion formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q \tag{5.3}$$

for the q-binomial coefficient (2.4), the explicit form of $h_n(x|q)$ is given by

$$h_n(\sinh\kappa\xi|q) = \sum_{k=o}^{n} (-1)^k q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathrm{e}^{(n-2k)\kappa\xi}. \tag{5.4}$$

The q-Hermite (2.3) and the q^{-1}-Hermite (5.4) polynomials are related to each other by the classical Fourier–Gauss transform [8]

$$H_n(\sin\kappa\xi|q)\mathrm{e}^{-\xi^2/2} = \imath^n \frac{q^{n^2/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_n(\sinh\kappa\eta|q)\mathrm{e}^{-\imath\xi\eta-\eta^2/2}\,\mathrm{d}\eta. \tag{5.5}$$

This means that the q^{-1}-Hermite functions

$$\psi_n(\eta|q^{-1}) = q^{n(n+1)/4} c_n(q) h_n(\sinh\kappa\eta|q)\mathrm{e}^{-\eta^2/2}, \tag{5.6}$$

obtained from (2.1) by the change $q \to q^{-1}$ of the parameter q, are connected with the q-Hermite functions (2.1) by the classical Fourier transform

$$\psi_n(\xi|q) = \frac{(\imath q^{-1/4})^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\imath\xi\eta} \psi_n(\eta|q^{-1})\,\mathrm{d}\eta. \tag{5.7}$$

Their expansion in terms of the Hermite functions (1.1) has the form

$$\psi_n(\eta|q^{-1}) = \sum_{k=0}^{\infty} C_{n,k}(q^{-1})\psi_k(\eta), \tag{5.8}$$

where the Fourier coefficients $C_{n,k}(q^{-1})$ are equal to

$$C_{n,k}(q^{-1}) = q^{n/4} \cos(n-k)\pi/2 C_{n,k}(q) \tag{5.9}$$

and the $C_{n,k}(q)$ are given in (4.5).

6. Relationship with the coherent states

In the study of a number of quantum-mechanical problems it turns out very useful to employ a system of coherent states. The wave functions of coherent states for the linear harmonic oscillator are expressed in terms of the Hermite functions (1.1) as

$$\psi(\xi; z) = \langle \xi | z \rangle = \exp(-|z|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \psi_n(\xi), \tag{6.1}$$

where z is the complex parameter. The q-Hermite functions (2.1) are in fact some linear combinations of $\psi(\xi; z)$ with particular values of the parameter z. Indeed, if one substitutes the explicit form of the Fourier coefficients (4.4) into (4.1), then the sum over the index k in it can be evaluated by (6.1). Thus the required relationship is

$$\psi_n(\xi|q) = \frac{\imath^n}{(q;q)_n^{1/2}} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \psi(\xi;\ \imath\sqrt{2}\kappa(k-n/2)). \tag{6.2}$$

In a similar manner, from (5.8) and (5.9) it follows that the corresponding relationship for the q^{-1}-Hermite functions (5.6) is

$$\psi_n(\eta|q^{-1}) = \frac{q^{n(n+1)/4}}{(q;q)_n^{1/2}} \sum_{k=0}^{n} (-1)^k q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q \psi(\eta; \sqrt{2}\kappa(n/2-k)). \tag{6.3}$$

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